Computer modelling of influence actuation and sensing in piezoelectric smart materials

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Piezoelectric materials create electrical charge when mechanically stressed and vice versa. In recent years, the range of applications for piezoelectric components has rapidly expanded in various fields. In the present work, experimental studies have been carried out to understand the behaviour of piezoelectric smart material and structure. Also finite element method based on numerical solution has been developed to analyse the deformation, electric potentials and natural frequencies of a piezoelectric smart structure subjected to external mechanical or electrical loadings. We considered eight-node hexahedral element for the analysis. We have developed a code to solve three-dimensional structures integrated with piezoelements using MATLAB software and the performance of the three-dimensional models has been compared with analytical methods for the same configuration. The standard benchmark problems have been considered for solving by presented finite element and analytical methods for the same configurations, and the comparison of results is in good agreement.

Key words: smart structures, piezoelectric, finite element, MATLAB

1. Introduction

Certain crystalline substances generating electrical charges under mechanical stress are called piezoelectric materials and conversely, if those crystals are placed in an electric field, they will experience mechanical strain. The first property makes them suitable as sensors, whereas the second property makes them suitable as actuators to control structural response. Because of these properties they can be used in measurements [1–6]. In recent years, piezoelectric materials have been

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integrated with structural systems to form a class of smart structures and the Finite Element Method can be used so that the number of iterations can be done to understand and optimise the process [7–10]. In the present work, formulation of a finite element has been done. The implementation of the above formulation has been carried out using MATLAB software. The developed code is validated using number of problems involving different materials such as isotropic, orthotropic and piezoelectric materials and their combination. Present solutions are compared with available literature values obtained from analytical, numerical and experimental approaches.

2. Constitutive equations

The basic constitutive equations for the linear theory of piezoelectricity are as follows [11]:

$$\begin{aligned} \{\sigma\} &= [C]\{\varepsilon\} - [d]\{E\}, \\ \{D\} &= \{d\}^{\mathrm{T}}\{\varepsilon\} + [b]\{E\}, \end{aligned} \tag{1}$$

where $\{\sigma\} = \{\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{23}, \sigma_{13}, \sigma_{12}\}^{\mathrm{T}}$ is the stress vector, $\{\varepsilon\} = \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{23}, \varepsilon_{13}, \varepsilon_{12}\}^{\mathrm{T}}$ the strain tensor, $\{E\} = \{E_1, E_2, E_3\}$ the electric field, $\{D\} = \{D_1, D_2, D_3\}$ the electric displacement or electric flux density vector, $[C]$ the elasticity constants matrix, $[b]$ the dielectric constants matrix, $[d]$ the piezoelectric coupling coefficients matrix or piezoelectric constants matrix.

3. Mechanical and electrical equilibrium equations

$$\delta\pi = \delta U - \delta W^{\mathrm{e}} = 0, \tag{2}$$

$$\delta U = \delta W^{\mathrm{e}}, \tag{3}$$

where δU and δW^{e} are virtual works of internal and external forces and

$$\int_V \delta\varepsilon\sigma \mathrm{d}V = \int_V \langle\delta\varepsilon\rangle\{\sigma\}\mathrm{d}V = \delta W^{\mathrm{e}}_{\mathrm{mech}} \tag{4}$$

and

$$-\int \delta E D \mathrm{d}V = \delta W_{\mathrm{elect}}. \tag{5}$$

4. Finite element modelling of piezoelectric material using 3D element

In the finite element formulation, the displacements u, v, w and the potentials are approximated as functions of the nodal displacements u_n and nodal potential ϕ_n, where n is node number of the element and the nodal shape functions N_i such that

$$\{u\} = [N_u]\{u_n\},$$
$$\{\phi\} = [N_\phi]\{\phi_n\}. \tag{6}$$

The electric field vector of an element is represented as follows:

$$\{E\} = \langle E_x, E_y, E_z\rangle^{\mathrm{T}}, \tag{7}$$

where

$$E_i = -\phi_i. \tag{8}$$

Consequently

$$\{\varepsilon\} = [B_u]\{u_n\},$$
$$\{E\} = -[B_\phi]\{\phi_n\}. \tag{9}$$

The matrices $[B_u]$ and $[B_\phi]$ contain the derivatives of the shape functions for the displacements and potentials which is written as follows:

$$[B_u] = \begin{bmatrix} \partial_x & 0 & 0 \\ 0 & \partial_y & 0 \\ 0 & 0 & \partial_z \\ 0 & \partial_z & \partial_y \\ \partial_z & 0 & \partial_x \\ \partial_y & \partial_x & 0 \end{bmatrix} [N_u], \tag{10}$$

$$[B_\phi] = \begin{bmatrix} \partial_x \\ \partial_y \\ \partial_z \end{bmatrix} [N_u]. \tag{11}$$

The external virtual work done by the external mechanical and electrical forces is

$$\delta W^{\mathrm{e}}_{\mathrm{mech}} = \langle \delta u_n\rangle\{F\},$$
$$\delta W^{\mathrm{e}}_{\mathrm{elect}} = \langle \delta \phi_n\rangle\{Q\}, \tag{12}$$

where $\{F\}$, $\{Q\}$ are external mechanical force and electrical charge vectors. Consequently, mechanical and electrical equilibrium equations can be written as follows:

$$\int \langle \delta u_n \rangle [B_u]^{\mathrm{T}} \left([c][B_u]\{u_n\} + [d][B_\phi]\{\phi_n\} \right) \mathrm{d}V = \langle \delta u_n \rangle \{F\} \tag{13}$$

and

$$\int \langle \delta \phi_n \rangle [B_\phi]^{\mathrm{T}} \left([d]^{\mathrm{T}}[B_u]\{u_n\} - [b][B_\phi]\{\phi_n\} \right) \mathrm{d}V = \langle \delta \phi_n \rangle \{Q\}, \tag{14}$$

or

$$\begin{aligned} \left(\int [B_u]^{\mathrm{T}}[c][B_u]\mathrm{d}V \right) \{u_n\} + \left(\int [B_u]^{\mathrm{T}}[d][B_\phi]\mathrm{d}V \right) \{\phi_n\} &= \{F\}, \\ \left(\int [B_\phi]^{\mathrm{T}}[d]^{\mathrm{T}}[B_u]\mathrm{d}V \right) \{u_n\} + \left(- \int [B_\phi]^{\mathrm{T}}[b][B_\phi]\mathrm{d}V \right) \{\phi_n\} &= \{Q\}. \end{aligned} \tag{15}$$

The stiffness matrices are defined as follows:

$$\begin{aligned} [K_{uu}] &= \int_V [B_u]^{\mathrm{T}}[c][B_u]\mathrm{d}V, \\ [K_{\phi\phi}] &= -\int_V [B_\phi]^{\mathrm{T}}[b][B_\phi]\mathrm{d}V, \\ [K_{\phi u}] &= \int_V [B_\phi]^{\mathrm{T}}[d]^{\mathrm{T}}[B_u]\mathrm{d}V, \\ [K_{u\phi}] &= [K_{\phi u}]^{\mathrm{T}}, \end{aligned} \tag{16}$$

$$\begin{aligned} [K_{uu}]\{u_n\} + [K_{u\phi}]\{\phi_n\} &= \{F\}, \\ [K_{\phi u}]\{u_n\} + [K_{\phi\phi}]\{\phi_n\} &= \{Q\}. \end{aligned} \tag{17}$$

Thus

$$\begin{bmatrix} [K_{uu}] & [K_{u\phi}] \\ [K_{\phi u}] & [K_{\phi\phi}] \end{bmatrix} \begin{Bmatrix} \{u_n\} \\ \{\phi_n\} \end{Bmatrix} = \begin{Bmatrix} \{F\} \\ \{Q\} \end{Bmatrix},$$

$$[K]\{u\} = \{F\}. \tag{18}$$

The above equations are included in formulation of an element to add the capability to solve static analysis of piezoelectric smart structures [5, 12–14].

Further the dynamic equations of a piezoelectric continuum can be derived from the Hamilton principle, in which the Lagrangian and the virtual work are properly adapted to include the electrical contributions as well as the mechanical ones. The potential energy density of a piezoelectric material includes contributions from the strain energy and from the electrostatic energy in the form

$$H = \frac{1}{2}[\{S\}^{\mathrm{T}}\{T\} - \{E\}^{\mathrm{T}}\{D\}]. \tag{19}$$

Similarly using the virtual work the related expression can be found

$$\delta W = \{u\}^{\mathrm{T}}\{F\} - \delta\Phi\sigma, \tag{20}$$

where $\{F\}$ is the external force and σ is the electric charge.

The variational principle governing the piezoelectric materials follows from the substitution of H and δW into the Hamilton principle:

$$\begin{aligned} 0 = \int_V & \big[\rho\{\delta u\}^{\mathrm{T}}\{\ddot{u}\} - \{\delta S\varepsilon\}^{\mathrm{T}}[c]\{\varepsilon\} + \{\delta\varepsilon\}^{\mathrm{T}}[d]^{\mathrm{T}}\{E\} \\ & + \{\delta E\}^{\mathrm{T}}[d]\{\varepsilon\} + \{\delta E\}^{\mathrm{T}}[b]\{E\} + \{\delta u\}^{\mathrm{T}}\{P_{\mathrm{b}}\}\big]\,\mathrm{d}V \\ & + \int_{\varepsilon_1}\{\delta u\}^{\mathrm{T}}\{P_{\mathrm{s}}\}\mathrm{d}\varepsilon + \{\delta u\}^{\mathrm{T}}\{P_{\mathrm{c}}\} - \int_{\varepsilon_2}\delta\Phi\sigma\mathrm{d}\varepsilon - \delta\Phi Q. \end{aligned} \tag{21}$$

The equation consists of finite element displacement field, electrical field and shape function of the finite element and arbitrary variation of the displacements $\{\delta u_i\}$ and electrical potentials $\{\delta\Phi_i\}$ compatible with the essential boundary conditions. Following Hamilton equation (7), the final dynamic equilibrium equations are written in the following form:

$$[M]\{\ddot{u}_i\} + [K_{uu}]\{u_i\} + [K_{u\Phi}]\{\Phi_i\} = \{f_i\}, \tag{22}$$

$$[K_{\Phi u}]\{u_i\} + [K_{\Phi\Phi}]\{\Phi_i\} = \{g_i\}, \tag{23}$$

with

$$[M] = \int_V \rho[N_u]^{\mathrm{T}}[N_u]\mathrm{d}V,$$

$$[K_{uu}] = \int_V [B_u]^{\mathrm{T}}[c][B_u]\mathrm{d}V,$$

$$[K_{u\Phi}] = \int_V [B_u]^{\mathrm{T}}[d]^{\mathrm{T}}[B_\Phi]\mathrm{d}V,$$

$$[K_{\Phi\Phi}] = -\int_V [B_\Phi]^{\mathrm{T}}[b][B_\Phi]\mathrm{d}V,$$

with

$$[K_{\Phi u}] = [K_{u\Phi}]^{\mathrm{T}}, \tag{24}$$

are the element mass, stiffness, piezoelectric coupling and capacitance matrices, respectively, and

$$\{f_i\} = \int_V [N_u]^{\mathrm{T}}\{P_{\mathrm{b}}\}\mathrm{d}V + \int_{S_1} [N_u]^{\mathrm{T}}\{P_{\mathrm{S}}\}\mathrm{d}S + [N_u]^{\mathrm{T}}\{P_{\mathrm{c}}\}, \tag{25}$$

$$\{g_i\} = -\int_{S_2} [N_\Phi]^{\mathrm{T}}\sigma\mathrm{d}S - [N_\Phi]^{\mathrm{T}}Q \tag{26}$$

are the external mechanical force and electric charge.

The above formulation is used for static and dynamic analysis of piezoelectric smart structure. The code has been developed using MATLAB.

5. Numerical investigations

The developed code has been validated using number of problems involving piezoelectric materials. The present solutions are compared with available literature values obtained from analytical, numerical and experimental approaches.

5.1 Piezo film subjected to shear load

A piezo film of 25.4 mm length, 25.4 mm width and a thickness 0.110 mm is rigidly fixed on the bottom surface. A force of 6.45 N is applied on the thickness cross-sectional area. The piezo film is considered to be PVDF. The piezo film is being stretched by the load applied, so as to produce a shear stress along the direction of applied load as shown in Fig. 1. The output voltage is determined.

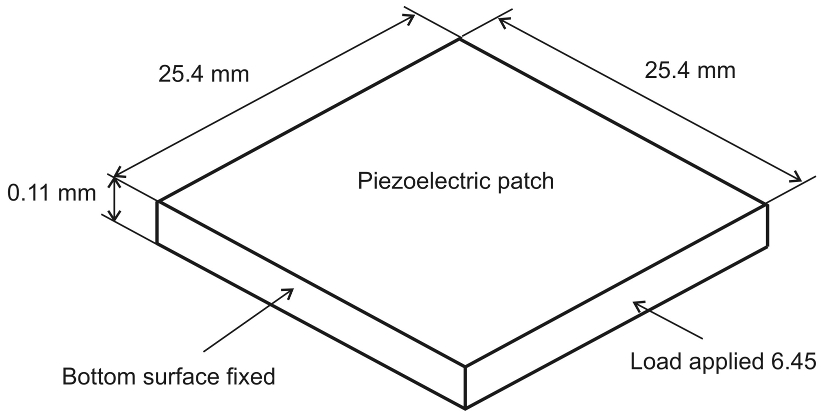

Fig. 1. Piezo film subjected to load.

5.1.1 Finite element analysis

The finite element model is shown in Fig. 2. Two elements are taken along the thickness direction and total 18 elements and 48 nodes are present in the model. The nodes corresponding to the bottom surface are fixed (i.e. $U_x = U_y = U_z = 0$). A shear load of 6.45 N is applied on the nodes along one of the edges. This results a maximum voltage near the bottom surface of the piezo film. In FEM, the values of results corresponding to the fourth degree of freedom are voltage values, which are determined.

The material properties required for various analyses are given in Table 1.

The material constants considered for isotropic materials are E and ν and for orthotropic materials E_{11}, E_{22}, E_{33}, G_{12}, G_{23}, G_{31}, ν_{12}, ν_{23}, and ν_{31}. The

Table 1. Material properties

Mechanical and electrical properties		Al alloy	PZT	PVDF
E_{11}, E_{22}, E_{33} [GPa]		68.9	65.0	2.0
G_{12}, G_{23}, G_{31} [GPa]		27.6	24.0	7.75
Piezoelectric constants $[\mathrm{C \cdot m^{-2}}]$	d_{31}	0	−6.5	0.03
	d_{32}	0	−6.5	0.03
	d_{33}	0	23.3	0.05
b_{11}, b_{22}, b_{33} $[\mathrm{F \cdot m^{-1}}]$		0	15.53E−9	106.2E−12
			15.53E−9	106.2E−12
			15.50E−9	105.2E−12
$\nu_{12}, \nu_{23}, \nu_{31}$		0.33	0.30	0.29
Density ρ $[\mathrm{kg \cdot m^{-3}}]$		2769	7600	1780

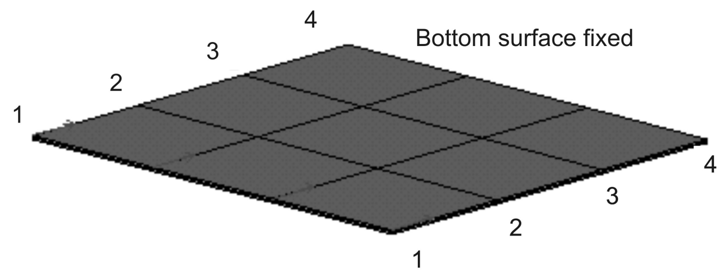

Fig. 2. Finite element of the piezo film.

piezoelectric material, which is assumed to be orthotropic apart from mechanical properties, the piezoelectric and permitivity constants required for the analysis are given in Table 1.

5.1.2 Analytical method

In sensor mode of application, shear load is applied and voltage is determined at piezoelectric element using the following equation (15):

$$V = \frac{g_{31} F}{w}, \tag{27}$$

where $g_{31} = 216 \times 10^{-3}\ \mathrm{V \cdot m \cdot N^{-1}}$, applied force $F = 6.45$ N, width $w = 2.54 \times 10^{-2}$ m, voltage $V = \dfrac{216 \times 10^{-3} \times 6.45}{2.54 \times 10^{-2}} = 54.8$ V.

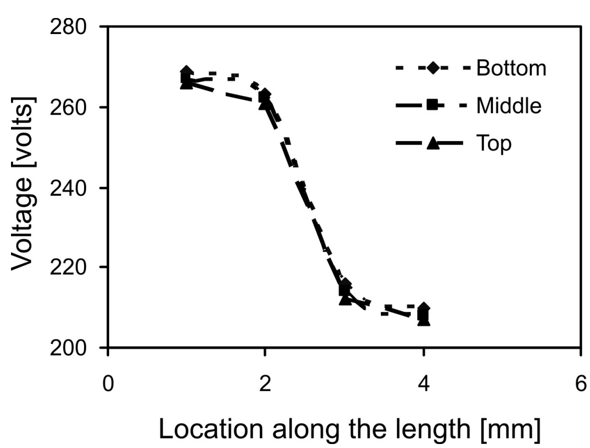

Fig. 3. Variation of voltage along the length.

Table 2. Comparison of voltage in piezo film

Sl. No.	Reference [15]	Number of elements	Present code
1	54.8 V	8	47 V
2		12	52 V
3		18	55.3 V

The voltage produced due to the applied load of 6.45 N is found to be 55.3 V from the present code and by theoretical results [15], which is 54.8 V.

The maximum output voltage developed across the film is compared with the theoretically calculated value and is shown in Table 2. The convergence analysis has been shown in Table 2. The results are found to be in good agreement.

The variation of voltage along the length is found at location marked as 1 to 4 (see Fig. 2) and is shown in Fig. 3.

5.2 Free vibration analysis of piezoelectric transducer

A piezoelectric transducer consists of a cube of PZT material with its polarisation direction aligned along the z-axis. Electrodes are placed on the two surfaces orthogonal to the polarisation axis. The first two coupled-mode (breathing-type

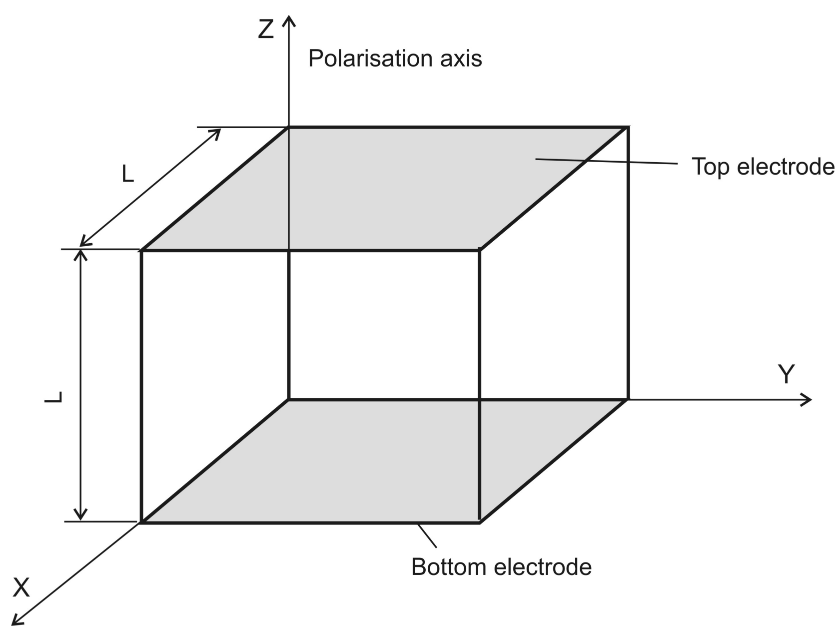

Fig. 4. Geometry of piezo transducer.

deformation) natural frequencies for the short circuit (resonance) are estimated. The PZT material properties are given in Table 1. The side of the cubic transducer is 0.02 m and the geometry of the transducer is shown in Fig. 4. A free vibration analysis of the transducer is carried out.

5.2.1 Finite element modelling and analysis

A piezoelectric transducer consists of a cube of PZT material. The electrode region represents equipotential surfaces, which are not modelled explicitly. For the short-circuit case the top and bottom electrodes are grounded (voltages are set equal to zero) and fixed at the base. Finite element model shown in Fig. 5 is created according to the given dimension. 64 elements and 125 nodes are present in the model from which natural frequencies are obtained.

Natural frequencies for the short circuit (resonance) case are determined and the results are as follows. The natural frequencies are obtained from the code and are compared with the experimental results given by [16]. The results are found to have good agreement as shown in Table 3.

Table 3. Frequencies in piezoelectric transducer

Sl. No.	Short circuit	Reference [16]	Present code
1	F1 kHz	66.70	65.66
2	F2 kHz	88.01	87.27

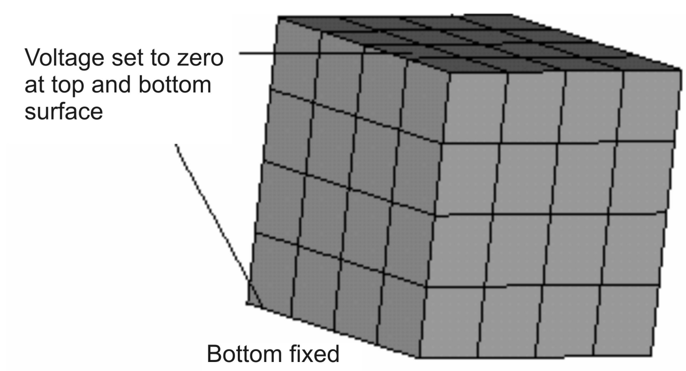

Fig. 5. FE model of piezoelectric transducer.

5.2.2 Analytical method

Analytical equations from [16] pertaining to the fundamental resonance frequency are given as follows:

$$F_z = \frac{1}{2L}\sqrt{\frac{1}{\rho \times S_{11}^{\mathrm{E}}}}, \tag{28}$$

where L is length, ρ is density, S_{11}^{E} is compliance.

The configuration of the present problem:
length $L = 0.02$ m, density $\rho = 7.5 \times 10^3$ kg $\cdot$ m^{-3}, compliance $S_{11}^{\mathrm{E}} = 18.5 \times 10^{-12}$ m$^2 \cdot$ N^{-1}.

The values of the present configuration are substituted in Eq. (28). The fundamental resonance frequency was estimated to be 66.7 kHz. It is clear that fundamental frequency obtained by the present code is in good agreement with the closed form solutions.

5.3 Smart beam subjected to voltage

This configuration has been taken up to understand the piezoelectric patch actuation on the aluminium beam and to verify the finite element code capability in solving the combined isotropic and piezoelectric material of the smart beam.

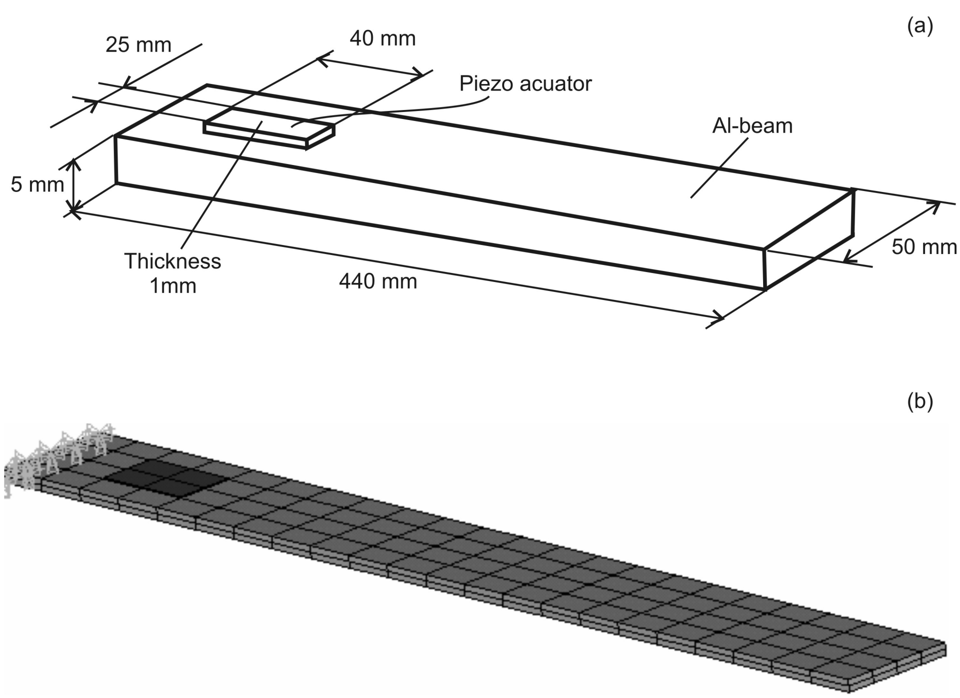

Fig. 6. Geometry of an aluminium beam with piezo actuator and finite element model: (a) geometry of the smart beam, (b) finite element model.

A thick aluminium beam is considered and the piezoelectric patch is mounted on this beam and subjected to electric field from 50 to 200 V. Geometry of the smart beam is shown in Fig. 6a, and its finite element model is shown in the Fig. 6b. The study of this case involves the combination of isotropic and piezoelectric materials.

5.3.1 Finite element analysis of the beam

The finite element model of the beam is shown in Fig. 6b. One end of the beam is fixed ($U_x = U_y = U_z = 0$) and the other one is set free to simulate the cantilever condition. The voltage is applied on piezo element from 50 to 200 V in steps of 50 V and the resulting tip deflection is determined. The convergence study is carried out by increasing the number of elements in finite element model and then an analysis is performed. The converged solution is obtained for 184 elements. These results are given in Table 4. The present FE results are compared with results obtained using closed form solution. The results of the two methods are given in Table 4. As can be seen from the table, they are in good agreement.

Table 4. Comparison of analytical and FE results for the smart beam

Sl. No.	Voltage applied [V]	Deflection [μm]			
		Analytical	Present code No. of elements		
			90	128	184
1	50	24	12	19	23
2	100	48	25	40	46
3	150	72	36	58	69
4	200	96	47	81	93

5.3.2 Analytical method

The deflection is determined by an analytical method using the equation

$$\delta = \left(\frac{m \cdot t_\mathrm{b}}{2I_\mathrm{b}E_\mathrm{b}}\right) \cdot l, \tag{29}$$

where δ is deflection, m is the moment generated due voltage applied is given by $m = \dfrac{6 \cdot d_{31} \cdot V}{t_\mathrm{c}t_\mathrm{b}} \cdot \dfrac{EI_\mathrm{b} \cdot EI_\mathrm{c}}{3EI_\mathrm{b} + 4EI_\mathrm{c}}$, I_b is moment inertia of the beam, I_c is moment inertia of the piezomaterial, d_{31} is piezoelectric constant, V is voltage applied, E_b is modulus of the beam, l is length of the beam, t_c is thickness of the piezomaterial, t_b is thickness of the beam, and

$$I_{\mathrm{b}} = \frac{w_{\mathrm{b}} t_{\mathrm{b}}^3}{12} = 520.833\ \mathrm{mm}^4, \tag{30}$$

$$I_{\mathrm{c}} = \frac{w_{\mathrm{c}} t_{\mathrm{c}}^3}{12} = 2.0833\ \mathrm{mm}^4. \tag{31}$$

Substituting the value of moment of inertia of the beam and piezo material from Eqs. (30) and (31) in (29) and also substituting other values related to the configuration of the problem, one can get the final displacement as follows:

For 50 V, substituting the values in the above Eq. (29):

$$\delta = \frac{6 \times 274 \times 10^{-9} \times 50 \times 70 \times 10^3 \times 520.83 \times 70 \times 10^3 \times 5 \times 440}{1 \times 5 \times 3 \times 70 \times 10^3 \times 520.83 + 4 \times 70 \times 10^3 \times 2.083 \times 2 \times 520.833 \times 83 \times 10^3} = 24\mu\mathrm{m}.$$

6. Conclusions

In the present work eight-node isoparametric finite element is developed for structural electromechanical analysis using MATLAB software. The formulation of eight-node isoparametric element is presented which can be applied for piezoelectric structures for structural and electromechanical analysis. The isotropic, orthotropic and piezoelectric materials are considered. The standard benchmark problems taken from literature are used to validate the present code. For all the cases considered, the analytical methods are used to solve the problem with the same configuration as that solved by presented finite element method. The results show good agreement with reference solutions.

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Received: 5.5.2008
Revised: 6.10.2008